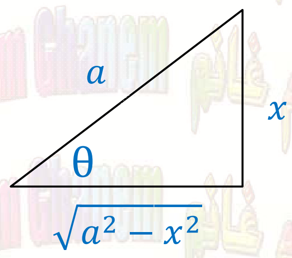
HOSSAM GHANEM

(21) 8.3 *Trigonometric Substitutions (A)*

Expression in integrand	$\sqrt{a^2 - x^2}$
Trigonometric substitution	$x = a\sin\theta \qquad dx = a\cos\theta \; d\theta$ $\sin\theta = \frac{x}{a}$ $\theta = \sin^{-1}\left(\frac{x}{a}\right)$



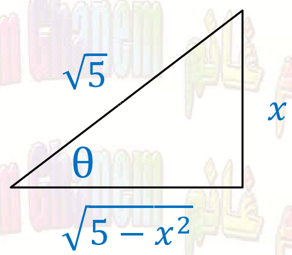
EXAMPLE

$$I = \int \frac{2}{x\sqrt{5 - x^2}}\, dx$$

$$x = \sqrt{5}\sin\theta \qquad dx = \sqrt{5}\cos\theta \; d\theta$$

$$\sin\theta = \frac{x}{\sqrt{5}}$$

$$\theta = \sin^{-1}\left(\frac{x}{\sqrt{5}}\right)$$



$$I = \int \frac{2}{\sqrt{5}\sin\theta\sqrt{5 - \left(\sqrt{5}\sin\theta\right)^2}}\,\sqrt{5}\cos\theta \; d\theta$$

$$= \int \frac{2\cos\theta}{\sin\theta\sqrt{5 - 5\sin^2\theta}}\; d\theta$$

$$= \int \frac{2\cos\theta}{\sin\theta\sqrt{5}\sqrt{1 - \sin^2\theta}}\, d\theta$$

$$= \frac{2}{\sqrt{5}}\int \frac{\cos\theta}{\sin\theta\sqrt{\cos^2\theta}}\, d\theta = \frac{2}{\sqrt{5}}\int \frac{\cos\theta}{\sin\theta\cos\theta}\, d\theta$$

$$= \frac{2}{\sqrt{5}}\int \csc\theta\, d\theta = \ln|\csc\theta - \cot\theta|$$

$$= \frac{2}{\sqrt{5}}\ln\left|\frac{\sqrt{5}}{x} - \frac{\sqrt{5 - x^2}}{x}\right| + C$$

Example 1 Evaluate the integral $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$

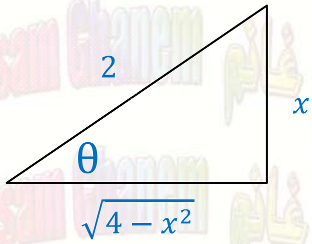
1 May 1994
25 December 2001
33 May 2004

Solution

$x = 2\sin\theta \qquad dx = 2\cos\theta \; d\theta$

$\sin\theta = \dfrac{x}{2}$

$\theta = \sin^{-1}\left(\dfrac{x}{2}\right)$



$$I = \int \frac{x^2}{\sqrt{4-x^2}}\,dx = \int \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}\cdot 2\cos\theta\;d\theta = \int \frac{4\sin^2\theta}{\sqrt{4(1-\sin^2\theta)}}\cdot 2\cos\theta\;d\theta =$$

$$= \int \frac{4\sin^2\theta}{\sqrt{4\cos^2\theta}}\cdot 2\cos\theta\;d\theta = \int \frac{4\sin^2\theta}{2\cos\theta}\cdot 2\cos\theta\;d\theta = 4\int \sin^2\theta\;d\theta$$

$$= \frac{4}{2}\int (1-\cos 2\theta)\;d\theta = 2\left(\theta - \frac{1}{2}\sin 2\theta\right) = 2\theta - \sin 2\theta + c = 2\theta - 2\sin\theta\cos\theta$$

$$= 2\sin^{-1}\left(\frac{x}{2}\right) - 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) + c = 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}x\sqrt{4-x^2} + c$$

Example 2 Evaluate the integral $\displaystyle\int \frac{1}{x^4\sqrt{16-x^2}}\,dx$

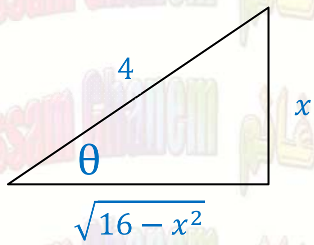
22 December 2000

Solution

$x = 4\sin\theta \qquad dx = 4\cos\theta \; d\theta$

$\sin\theta = \dfrac{x}{4}$

$\theta = \sin^{-1}\left(\dfrac{x}{4}\right)$



$$I = \int \frac{1}{x^4\sqrt{16-x^2}}\,dx = \int \frac{4\cos\theta}{4^4\ \sin^4\theta\sqrt{16-16\sin^2\theta}}\,d\theta = \int \frac{4\cos\theta}{4^4\ \sin^4\theta\cdot 4\cos\theta}\,d\theta = \frac{1}{4^4}\int \frac{1}{\sin^4\theta}\,d\theta$$

$$= \frac{1}{4^4}\int \csc^4\theta\;d\theta = \frac{1}{4^4}\int \csc^2\theta\;\csc^2\theta\,d\theta = \frac{1}{4^4}\int (1+\cot^2\theta)\;\csc^2\theta\,d\theta$$

Let $u = \cot\theta \qquad du = -\csc^2\theta\,d\theta$

$$\therefore I = \frac{-1}{4^4}\int (1+u^2)\;du = \frac{-1}{4^4}\left[u + \frac{1}{3}\,u^3\right] + c$$

$$= \frac{-1}{4^4}\left[\cot\theta + \frac{1}{3}\cot^3\theta\right] + c = \frac{-1}{4^4}\left[\frac{\sqrt{16-x^2}}{x} + \frac{1}{3}\left(\frac{\sqrt{16-x^2}}{x}\right)^3\right] + c$$

Example 3 50 Dec. 15, 2009

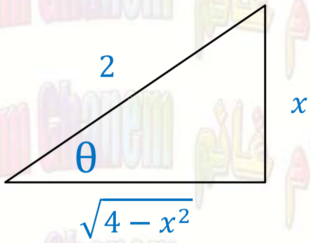
Evaluate the following integral : $\int x^2\sqrt{4-x^2}\,dx$ ($3\frac{1}{2}$ points)

Solution

$x = 2\sin\theta$ $dx = 2\cos\theta\ \ d\theta$

$\sin\theta = \frac{x}{2}$

$\theta = \sin^{-1}\left(\frac{x}{2}\right)$



$$I = \int x^2\sqrt{4-x^2}\,dx = \int 4\sin^2\theta\sqrt{4-4\sin^2\theta}\cdot 2\cos\theta\ \ d\theta = \int 4\sin^2\theta\cdot 2\cos\theta\cdot 2\cos\theta\ \ d\theta$$

$$= 4\int 4\sin^2\theta\cos^2\theta\ \ d\theta = 4\int(2\sin\theta\cos\theta)^2 d\theta = 4\int\sin^2 2\theta\,d\theta = 2\int(1-\cos 4\theta)d\theta$$

$$= 2\left(\theta - \frac{1}{4}\sin 4\theta\right) + c = 2\left(\theta - \frac{1}{2}\sin 2\theta\cos 2\theta\right) + c = 2[\theta - \sin\theta\cos\theta\,(1-2\sin^2\theta)] + c$$

$$= 2\left[\sin^{-1}\left(\frac{x}{2}\right) - \left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right)\left(1-2\left(\frac{x}{2}\right)^2\right)\right] + c$$

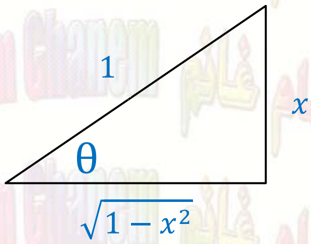
Example 4 Evaluate the following integral $\int \frac{1-\sqrt{1-x^2}}{x^2}\,dx$ 35 January 24, 2010

Solution

$x = \sin\theta$ $dx = \cos\theta\ \ d\theta$

$\sin\theta = \frac{x}{1}$

$\theta = \sin^{-1}(x)$



$$I = \int \frac{1-\sqrt{1-x^2}}{x^2}\,dx = \int \frac{1-\sqrt{1-\sin^2\theta}}{\sin^2\theta}\cos\theta\ \ d\theta = \int\frac{1-\cos\theta}{\sin^2\theta}\cos\theta\ \ d\theta$$

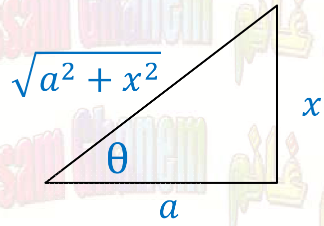
$$= \int\frac{\cos\theta-\cos^2\theta}{\sin^2\theta}\ \ d\theta = \int\frac{\cos\theta}{\sin^2\theta} - \frac{\cos^2\theta}{\sin^2\theta}\ \ d\theta = \int(\cot\theta\csc\theta - \cot^2\theta)\,d\theta$$

$$= \int(\cot\theta\csc\theta - \csc^2\theta + 1)\,d\theta = -\csc\theta + \cot\theta + \theta + c$$

$$= -\frac{1}{x} + \frac{\sqrt{1-x^2}}{x} + \sin^{-1}(x) + c$$

HOSSAM GHANEM

(21) 8.3 *Trigonometric Substitutions (A)*

Expression in integrand	$\sqrt{a^2+x^2}$ & $\sqrt{x^2+a^2}$
Trigonometric substitution	$x = a\tan\theta$ $\quad dx = a\sec^2\theta\ d\theta$ $\tan\theta = \frac{x}{a}$ $\theta = \tan^{-1}\left(\frac{x}{a}\right)$ 

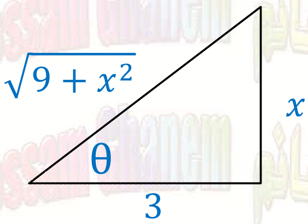
EXAMPLE

$$I = \int \frac{1}{\sqrt{9+x^2}}\,dx$$

$x = 3\tan\theta \qquad dx = 3\sec^2\theta\ d\theta$

$\tan\theta = \frac{x}{3}$

$\theta = \tan^{-1}\left(\frac{x}{3}\right)$



$$I = \int \frac{1}{\sqrt{9+(3\tan\theta)^2}}\,3\sec^2\theta\ d\theta$$

$$= \int \frac{3\sec^2\theta}{\sqrt{9+9\tan^2\theta}}\,d\theta = \int \frac{3\sec^2\theta}{3\sqrt{1+\tan^2\theta}}\,d\theta$$

$$= \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\,d\theta = \int \frac{\sec^2\theta}{\sec\theta}\,d\theta = \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C$$

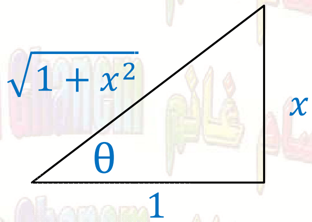
Example 5 Evaluate the following integral $\int \frac{1}{x^2\sqrt{x^2+1}}\,dx$ 18 December 1999

Solution

$x = \tan\theta \qquad dx = \sec^2\theta\ d\theta$

$\tan\theta = \frac{x}{1}$

$\theta = \tan^{-1}(x)$



$$I = \int \frac{1}{x^2\sqrt{x^2+1}}\,dx = \int \frac{1}{\tan^2\theta\sqrt{\tan^2\theta+1}}\cdot\sec^2\theta\ d\theta = \int \frac{\sec^2\theta}{\tan^2\theta\sqrt{\sec^2\theta}}\,d\theta == \int \frac{\sec^2\theta}{\tan^2\theta\sec\theta}\,d\theta$$

$$= \int \frac{\sec\theta}{\tan^2\theta}\ d\theta = \int \frac{1}{\tan^2\theta}\cdot\sec\theta\ \ d\theta = \int \frac{\cos^2\theta}{\sin^2\theta}\cdot\frac{1}{\cos\theta}\,d\theta = \int \frac{\cos\theta}{\sin^2\theta}\ d\theta$$

$$= \int \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin\theta}\ d\theta = \int \cot\theta\csc\theta\ \ d\theta = -\csc\theta + c = -\frac{\sqrt{x^2+1}}{x} + c$$

Example 6 * Evaluate the following integrals. $\int \frac{1}{x\sqrt{1+\sqrt{x}}}\,dx$ (3 pts)

55 July 23 , 2011

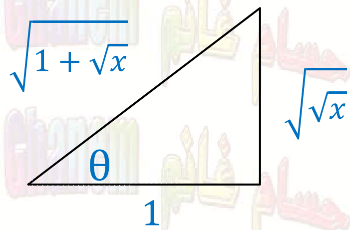
Solution

$\sqrt{x} = \tan^2\theta \qquad \therefore x = \tan^4\theta \qquad \therefore dx = 4\tan^3\theta\sec^2\theta\ d\theta$

$\sqrt{\sqrt{x}} = \tan\theta \qquad dx = 4\tan^3\theta\sec^2\theta\ d\theta$

$\tan\theta = \frac{\sqrt{\sqrt{x}}}{1}$

$\theta = \tan^{-1}\left(\sqrt{\sqrt{x}}\right)$



$$I = \int \frac{1}{x\sqrt{1+\sqrt{x}}}\,dx = \int \frac{1}{\tan^4\theta\ \sqrt{1+\tan^2\theta}}\cdot 4\tan^3\theta\sec^2\theta\ d\theta = 4\int \frac{\sec^2\theta}{\tan\theta\sec\theta}\ d\theta$$

$$= 4\int \frac{\sec\theta}{\tan\theta}\ d\theta = 4\int \frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}\ d\theta = 4\int \frac{1}{\sin\theta}\ d\theta = 4\int \csc\theta\ \ d\theta = 4\ln|\csc\theta - \cot\theta| + c$$

$$= 4\ln|\csc\theta - \cot\theta| + c = 4\ln\left|\frac{\sqrt{1+\sqrt{x}}}{\sqrt{\sqrt{x}}} - \frac{1}{\sqrt{\sqrt{x}}}\right| + c$$

Homework

1	Evaluate the integral	$\int \frac{2-x^2}{\sqrt{1-x^2}}\,dx$	
2	Evaluate the integral	$\int \frac{x^2}{\sqrt{25-x^2}}\,dx$	14 Nov. 1998
3	Evaluate the integral	$\int \frac{1}{x^2(1+x^2)^{\frac{5}{2}}}\,dx$	42 December 2006 A
4	Evaluate the integral	$\int \frac{3^x}{\sqrt{1+9^x}}\,dx$	
5	(4 pts.) Evaluate the following integral	$\int x^2\sqrt{1-x^2}\ dx$	36 June 6, 2010
6	(5 pts.) Evaluate	$\int \frac{x^2dx}{\sqrt{1-x^2}+\sqrt{1+x^2}}$	38 Jan. 22, 2011
7	Evaluate the following integral	$\int \frac{(1-x^2)^{\frac{3}{2}}}{x}\ dx$	14 June 4 , 2011
8	Evaluate the following integral	$\int \sqrt{1-x^2}\,\sin^{-1}x\,dx$	
9	Evaluate the integral	$\int \sqrt{1+\sqrt{x}}\,dx$	32 December 2003
10	Evaluate the following ($2\,{}^1/_2$ points)	$\int \frac{1}{x^2\sqrt{25-x^2}}\,dx$	56 11 December 2011

Example 1

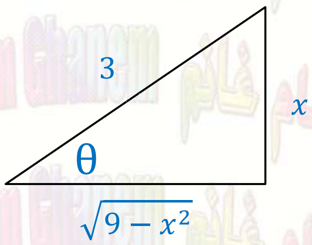
Evaluate the following integral $\int_{-1}^{1} \frac{1}{\sqrt{9-x^2}}\, dx$

Solution

$x = 3\sin\theta$ $\quad dx = 3\cos\theta \;\; d\theta$

$\sin\theta = \frac{x}{3}$

$\theta = \sin^{-1}(x)$



$$I = \int_{-1}^{1} \frac{1}{\sqrt{9-x^2}}\, dx$$

$$I_1 = \int \frac{1}{\sqrt{9-x^2}}\, dx = \int \frac{3\cos\theta}{\sqrt{9-9\sin^2\theta}}\, d\theta = \int \frac{3\cos\theta}{3\sqrt{1-\sin^2\theta}}\, d\theta = \int \frac{3\cos\theta}{3\cos\theta}\, d\theta = \int d\theta = \theta + c$$

$$= \sin^{-1}\frac{x}{3} + c$$

$$I = \left[\sin^{-1}\frac{x}{3}\right]_{-1}^{1} = \sin^{-1}\frac{1}{3} - \sin^{-1}\frac{-1}{3} = \sin^{-1}\frac{1}{3} + \sin^{-1}\frac{1}{3} = 2\sin^{-1}\frac{1}{3}$$

Example 3

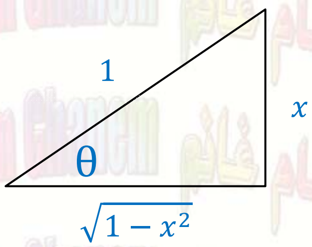
Evaluate the integral $\int \sqrt{1-x^2}\,\sin^{-1} x\, dx$

Solution

$x = \sin\theta$ $\quad dx = \cos\theta \;\; d\theta$

$\sin\theta = \frac{x}{1}$

$\theta = \sin^{-1} x$



$$I = \int \sqrt{1-x^2}\,\sin^{-1} x\, dxI = \int \sqrt{1-\sin^2\theta} \cdot \theta \cdot \cos\theta \;\; d\theta$$

$$= \int \theta \cdot \cos^2\theta \;\; d\theta == \int \theta \cdot \frac{1}{2}(1+\cos 2\theta) \;\; d\theta$$

$u = \theta \qquad dv = \frac{1}{2}\,(1+\cos 2\theta)d\theta$

$du = d\theta \qquad v = \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)$

$$I = uv - \int v\, du$$

$$I = \frac{\theta}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) - \frac{1}{2}\int \left(\theta + \frac{1}{2}\sin 2\theta\right) \;\; d\theta = \frac{\theta}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right) - \frac{1}{2}\left(\frac{1}{2}\theta^2 - \frac{1}{4}\cos 2\theta\right) + c$$

$$= \frac{\theta}{2}(\theta + \sin\theta\cos\theta) - \frac{1}{2}\left(\frac{1}{2}\theta^2 - \frac{1}{4}(1 - 2\sin^2\theta)\right) + c$$

$$= \frac{1}{2}\sin^{-1}x\left(\sin^{-1}x + x\sqrt{1-x^2}\right) - \frac{1}{2}\left[\frac{1}{2}(\sin^{-1}x)^2 - \frac{1}{4}(1-2x^2)\right] + c$$

Example 4

Evaluate the integral $\displaystyle\int \frac{(\ln x)^3}{x\sqrt{4-(\ln x)^4}}\,dx$ 43 May 2007 A

Solution

$$I = \int \frac{(\ln x)^3}{x\sqrt{4-(\ln x)^4}}\,dx$$

Let $t = 4-(\ln x)^4$ $\quad dt = -4(\ln x)^3 \cdot \frac{1}{x}\,dx$ $\quad \frac{-1}{4}\,dt = \frac{(\ln x)^3}{x}\,dx$

$$I = \int \frac{(\ln x)^3}{x\sqrt{4-(\ln x)^4}}\,dx = \int \frac{(\ln x)^3}{x}\cdot\frac{1}{\sqrt{4-(\ln x)^4}}\,dx = \frac{-1}{4}\int \frac{1}{\sqrt{u}}\,du = \frac{-1}{2}\sqrt{u} + c$$

$$= \frac{-1}{2}\sqrt{4-(\ln x)^4} + c$$

Example 6

Evaluate the integral $\displaystyle\int \sqrt{1+\sqrt{x}}\,dx$ 32 December 2003

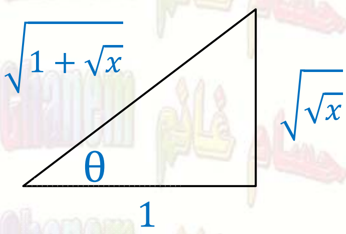
Solution

$\sqrt{x} = \tan^2\theta$ $\quad \therefore x = \tan^4\theta$ $\quad \therefore dx = 4\tan^3\theta\sec^2\theta\,d\theta$

$\sqrt{\sqrt{x}} = \tan\theta$ $\quad dx = 4\tan^3\theta\sec^2\theta\,d\theta$

$$\tan\theta = \frac{\sqrt{\sqrt{x}}}{1}$$

$$\theta = \tan^{-1}\left(\sqrt{\sqrt{x}}\right)$$



$$I = \int \sqrt{1+\sqrt{x}}\,dx = \int \sqrt{1+\tan^2\theta}\cdot 4\tan^3\theta\sec^2\theta\,d\theta = \int \sec\theta\cdot 4\tan^3\theta\sec^2\theta\,d\theta$$

$$= 4\int \tan^3\theta\sec^3\theta\,d\theta = 4\int \tan^2\theta\sec^2\theta\cdot\tan\theta\sec\theta\,d\theta$$

$$= 4\int (\sec^2\theta - 1)\sec^2\theta\cdot\tan\theta\sec\theta\,d\theta$$

Let $t = \sec\theta$ $\quad \rightarrow dt = \sec\theta\tan\theta\,d\theta$

$$I = 4\int (t^2-1)\,t^2\,dt = 4\int (t^4 - t^2)\,dt = \frac{4}{5}t^5 - \frac{4}{3}t^3 + c$$

$$= \frac{4}{5}(\sec\theta)^5 - \frac{4}{3}(\sec\theta)^3 + c$$

$$= \frac{4}{5}\left(\sqrt{1+\sqrt{x}}\right)^5 - \frac{4}{3}\left(\sqrt{1+\sqrt{x}}\right)^3 + c$$